Schemes 2018 Exercise 2

Question 1. Let S be a finitely generated multiplicative system $S =< f_1, .., f_k >$ in a ring R. Let $\phi_S : R \to S^{-1}R$ denote the map given by $\phi_S(r) = r/1$. Prove that $\phi_S^* : spec(S^{-1}R) \to spec(R)$ is an open embedding (i.e. it is a continuous open injective map), with image $D(f_1 \cdot ... \cdot f_k)$.

Question 2. Let $\mathfrak{p} = (p, q(x))$ be a maximal ideal of $\mathbb{Z}[x]$. Show that $(f) \subseteq \mathfrak{p}$ if and only if the reduction $\bar{f}(x) \in \mathbb{F}_p[x]$ is divisible by $q(x)$.

Question 3. Let A be a domain. Then $\overline{\{0\}} = spec(A)$.

Question 4. Let X be a topological space:

- If $Y \subseteq X$ is irreducible, so is $\overline{Y}$.
- For continuous $\phi : X \to Z$, $Y \subseteq X$ is irreducible $\implies$ $\phi(Y)$ is irreducible (both with the induced topologies from X, Z respectively).

Question 5. Prove that A is Noetherian $\implies$ $spec(A)$ is Noetherian. Find a counter example for the converse.

Question 6. Prove that A is Noetherian $\implies$ $S^{-1}A$ is Noetherian for every multiplicative system S.

Question 7. * Let A be a ring. Let $B \subseteq A$ be a subring.

- Is it true that if A is finitely generated then so is B ?
- Is it true that if A is Noetherian then so is B ?